

A new construction for spinor wave equations

S. de Toro Arias^{1,2,a} and Ch. Vanneste^{1,b}

¹ Laboratoire de Physique de la Matière Condensée^c, Université de Nice-Sophia Antipolis, Parc Valrose, B.P. 71, 06108 Nice Cedex 02, France

² Service de Physique de l'État Condensé, Centre d'Études de Saclay, l'Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France

Received: 3 November 1997 / Received in final form: 9 February 1998 / Accepted: 16 February 1998

Abstract. The construction of discrete scalar wave propagation equations in arbitrary inhomogeneous media was recently achieved by using elementary dynamical processes realizing a discrete counterpart of the Huygens principle. In this paper, we generalize this approach to spinor wave propagation. Although the construction can be formulated on a discrete lattice of any dimension, for simplicity we focus on spinors living in $1 + 1$ space-time dimensions. The Dirac equation in the Majorana-Weyl representation is directly recovered by incorporating appropriate symmetries of the elementary processes. The Dirac equation in the standard representation is also obtained by using its relationship with the Majorana-Weyl representation.

PACS. 03.65.Pm Relativistic wave equations – 02.70.-c Computational techniques

1 Introduction

The formulation of wave propagation by using the Huygens principle was investigated some years ago by the Transmission Line Matrix Modeling method (TLM) [1]. This method retrieves the Maxwell equations by introducing current and voltage impulses which propagate along the bonds and are scattered on the nodes of a mesh of transmission lines. Refinements lead the TLM method to describe complex boundary conditions, gain or losses and propagation in inhomogeneous media for electromagnetic waves [2]. Such an idea was renewed recently for studying time-dependent wave propagation for scalar waves in inhomogeneous media [3, 4]. Instead of considering currents or voltage pulses, the authors introduced some arbitrary scalar quantities that propagate on a Cartesian lattice. Both approaches include the action-by-proximity of the pulses, or of the scalar quantities, when they propagate from node to node in one time step and the emission of secondary waves at each node by means of a scattering process which emits the incident energy in all directions. Since voltage and current impulses are equivalent to electric and magnetic fields on a two-dimensional mesh, it is not surprising that the TLM method applies to the Maxwell equations. Nevertheless, the nature of the wave equations which result from such a formulation applied to scalar quantities propagating on a Cartesian lattice remains unclear. A beginning of answer arises in the work of

Sornette *et al.* [5], who have enlarged the scalar model to construct the Klein-Gordon equation and the Schrödinger equation. However, the results were limited to homogeneous media and the Schrödinger equation was ill-defined in the continuum limit. In order to generalize this attempt, a constructive approach was explored recently [6] to recover various kinds of scalar waves in an inhomogeneous medium. Starting from basic principles and incorporating fundamental symmetries to the scattering nodes, the authors derived in a systematic way time-dependent scalar wave equations in inhomogeneous media. They exhibited a unified equation which properly tuned by a unique parameter yields either the Klein-Gordon equation or the Schrödinger equation with a well defined continuum limit. This derivation offers a general framework, including the related TLM approach and opens up possible generalizations to describe spinor or vector wave propagation.

In the present paper, we extend this previous work to the description of discrete wave propagation equations for free spinor fields. For simplicity, we restrict the presentation to the derivation of the Dirac equation on a one-dimensional regular lattice but the proof can be worked out on any underlying discrete lattice. Throughout the paper, we call currents the scalar quantities which obey a simple dynamics of propagation and scattering. The problem being linear, we define naturally the two scalar components of the spinor field as linear superpositions of those currents. In order to derive coupled propagation equations linking together both components of the spinor field, we first require closure conditions compelling the form of the matrices which describe the scattering processes. Finally, by taking into account appropriate space-time relativistic

^a e-mail: sdetoro@spec.saclay.cea.fr

^b e-mail: vanneste@ondine.unice.fr

^c CNRS UMR 6622

symmetries, the discretized Dirac equation in $1 + 1$ dimensions is recovered. The current model is constructed for the two usual representations of the Dirac equation in $1 + 1$ space-time dimension, *i.e.* the Majorana-Weyl representation and the standard representation.

Independently from the above approach, different kinds of microscopic models describing spinor wave equations have been proposed in the literature. Those models belong to two distinct classes: they are either based on a random walk description, leading to Euclidean-invariant spinor wave equations, or they rely on a complex hopping-type dynamics on a lattice, leading to relativistic-invariant spinor wave equations. The first class of microscopic probabilistic models deals with critical behavior of the free Majorana spinor field where the time variable t is turned into a parameter that controls the approach to a critical point (the system is at criticality when $t \rightarrow \infty$). It is indeed well known [7,8] that local scalar field theories can be studied within a random walk representation, where the time t corresponds to the length of a brownian path. Among such approaches, Mc Keon *et al.* [9] have constructed a probabilistic model, based on binomial processes, to describe the spinor field in $1 + 1$ space-time dimensions. As the Dirac equation is recovered from real stochastic processes, their method applies only to the real Majorana representation of the Dirac equation. Another random walk approach, including a spin factor, is developed in [10]. The model is defined on three-dimensional lattices and exhibits different critical behavior depending on the value of the spin. The second class of models deals with the Dirac equation itself. The paradigm of those lattice models are the Susskind fermions [11] which have been extensively used in lattice gauge theories. A different but closed approach is exposed in [12], where the model describes the dynamics of point-like spinless particles on a three-dimensional square lattice in the presence of complex hopping rates of modulus unity. Related works have been applied to study condensed matter topics such as the ground state of the Heisenberg anti-ferromagnet in two or three dimensions [13], or the energy spectrum of flux states [14,15]. The construction in terms of currents presented in this paper can be considered as belonging to that second class of models.

The backbone of the paper are Sections 2 and 3, where we describe the construction of the Dirac equation in the Majorana-Weyl and standard representations respectively. Sections 2.1 and 3.1 introduce the basis of the current model in both representations. Sections 2.2, 3.2, deal with the derivation of the discrete coupled propagation equations. The Majorana-Weyl construction turns out to be solvable by choosing suitable symmetries in Section 2.3. The current model is achieved by computing the scattering matrices in Section 2.4. On the contrary, the standard representation cannot be constructed in the same direct way. Hence, to compute the scattering matrices we use the linear transformation linking both representations (Sect. 3.3). Finally, Section 4 is devoted to discuss further issues and applications of this work.

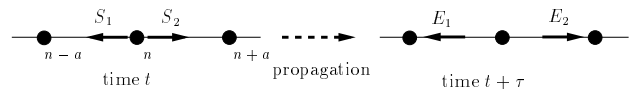


Fig. 1. Sketch of the propagation step. The outgoing currents S_i , $i = 1, 2$, on node n propagate in one time step to become incident currents E_j , $j = 1, 2$, on the neighbor nodes.

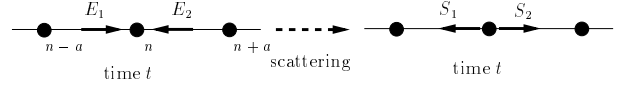


Fig. 2. Sketch of the scattering step. The incident currents E_i , $i = 1, 2$, on node n are scattered instantaneously to become outgoing currents S_j , $j = 1, 2$.

2 Majorana-Weyl representation

2.1 Basic definitions: currents and fields

This section is devoted to the construction of the Dirac equation in the Majorana-Weyl representation, which reads (see [16])

$$i \frac{\partial \Psi}{\partial t} = \left[-ic\sigma_3 \frac{\partial}{\partial x} + \frac{mc^2}{\hbar} \sigma_1 \right] \Psi, \quad (1)$$

where $\Psi = (\psi_L \ \psi_R)^T$ is the two-component spinor field and σ_i , $i = 1, 2, 3$, denotes the Pauli matrices.

The elementary bricks of the construction are called currents. The currents are complex numbers which propagate from node to node along the bonds of a Cartesian lattice, each bond carrying two currents propagating in opposite directions. For simplicity, the lattice is chosen to be one-dimensional and regular with a lattice mesh size a . At any time t , the system is completely defined by the values of all currents along the chain. The time variable is also discrete and τ denotes the time unit. A current propagates between two neighboring nodes in one time step. All the currents propagate simultaneously, *i.e.* the outgoing currents, denoted S_i , $i = 1, 2$, leave the nodes at some time t and become incident currents on the neighboring nodes at time $t + \tau$. The incident currents are denoted E_i , $i = 1, 2$ (Fig. 1).

After this propagation step, all the incident currents are instantaneously scattered. The scattering process is described by scattering matrices attached to each node. It transforms the incident currents E_i , $i = 1, 2$, on one node into outgoing currents S_j , $j = 1, 2$ on the same node (Fig. 2).

Since our aim is to describe the two chiral fields ψ_L and ψ_R , two kinds of currents, L and R currents, are naturally introduced, leading to four currents propagating simultaneously on each bond. For clarity, the one-dimensional lattice can be pictured by two sublattices carrying the L and R currents respectively (Fig. 3). Due to the structure of the Dirac equation (1) which displays a coupling term between the two chiral components ψ_L and ψ_R , we also introduce additional currents which propagate between the two sublattices in one time step as depicted in Figure 4.

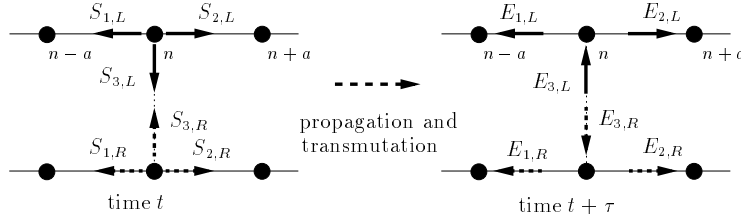


Fig. 4. Propagation step for the L and R currents.

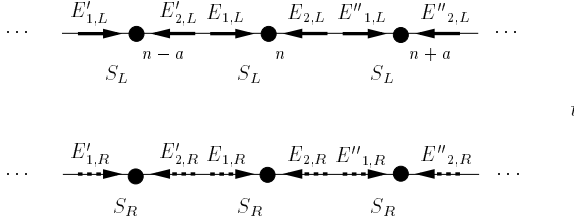


Fig. 3. The chain is split into two subchains. The upper and lower subchains carry the L and R currents respectively.

These additional currents, which are called the L or R node-current in the following, do not propagate along the chain but remain located on the same node. They can be considered as propagating along an internal time axis which couples the L and R sublattices at each node of the chain. Moreover, these node currents participate in the same scattering process as the propagating currents (Fig. 5).

Thus, the scattering process reads

$$S_{i,L}(n, t) = \sum_{j=1}^3 s_{ij,L} E_{j,L}(n, t), \quad i = 1, 2, 3, \quad (2)$$

$$S_{k,R}(n, t) = \sum_{l=1}^3 s_{kl,R} E_{l,R}(n, t), \quad k = 1, 2, 3, \quad (3)$$

where $s_{ij,L}$ and $s_{kl,R}$ are the complex entries of the scattering matrices S_L and S_R attached to each node. The ensemble of all the scattering matrices belonging to the chain defines the medium or the “background” in which the currents live. Once the scattering matrices are known, the evolution of the system at any time is completely determined given any initial values of the currents. Throughout the paper the scattering matrices are chosen to be identical from node to node indicating that the medium is uniform. However, the matrices S_L and S_R do not need to be identical.

The dynamics includes all the features of a discrete counterpart of the well-known Huygens principle. Namely, the principle of action-by-proximity from one node to the nearest-neighboring nodes is taken into account by the propagation step, while the interferences and the radiation of the energy in all directions are taken into account by the scattering process.

Let us introduce some convenient notations. The bonds attached to a given node are numbered as displayed in

Figure 5: 1 for the left bond, 2 for the right bond and 3 for the node bond.

Now consider a current $S_k(n, t)$ leaving the node n along the bond k , $k = 1, 2, 3$ (Fig. 6). At the next time step, this current becomes an input current E on the neighbor node along the k th direction. This neighbor node is labelled n_k . Note that this notation implies $n_{\bar{3}} = n$. Moreover, viewed from node n_k , the bond on which lies the input current E is denoted \bar{k} , leading to labelling this current as $E = E_{\bar{k}}(n_k, t + \tau)$. This notation implies $\bar{1} = 2$, $\bar{2} = 1$, $\bar{3} = 3$. Hence, the relations linking the outgoing and the incident currents during the propagation step read

$$E_{k,L}(n, t + \tau) = S_{\bar{k},L}(n_k, t), \quad k = 1, 2, \quad (4)$$

$$E_{k,R}(n, t + \tau) = S_{\bar{k},R}(n_k, t), \quad k = 1, 2, \quad (5)$$

$$E_{3,L}(n, t + \tau) = S_{3,R}(n, t), \quad (6)$$

$$E_{3,R}(n, t + \tau) = S_{3,L}(n, t). \quad (7)$$

The propagation rules (5) take into account a transmutation of the node currents $S_{3,L}$ and $S_{3,R}$ into $E_{3,R}$ and $E_{3,L}$ respectively.

To pursue the construction of the Dirac equation within this two-species current model, the chiral fields need to be defined. The fields ψ_L , ψ_R are complex valued functions which are defined on the nodes of the chain. The linearity of the problem compels the choice for each chiral field ψ_L or ψ_R to be a linear superposition of the L or R currents respectively. As the outgoing currents can be expressed as a function of the incident currents according to the scattering equations (2, 3) it is sufficient to define the chiral fields in terms of the incident currents solely:

$$\psi_L(n, t) = \sum_{k=1}^3 \lambda_{k,L} E_{k,L}(n, t), \quad (8)$$

$$\psi_R(n, t) = \sum_{l=1}^3 \lambda_{l,R} E_{l,R}(n, t), \quad (9)$$

where $\lambda_{k,L}$, $\lambda_{l,R}$ are complex numbers.

2.2 Discrete propagation equations linking ψ_L and ψ_R

Under certain assumptions, which we call thereafter closure conditions, we show that the two chiral fields, ψ_L ,

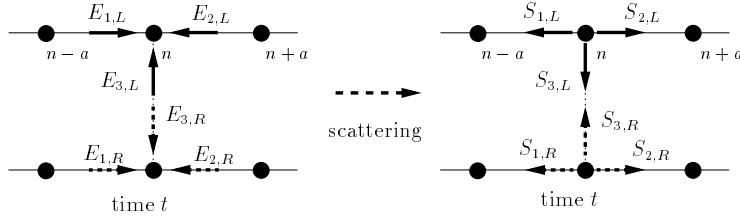


Fig. 5. Scattering step for the L and R currents.

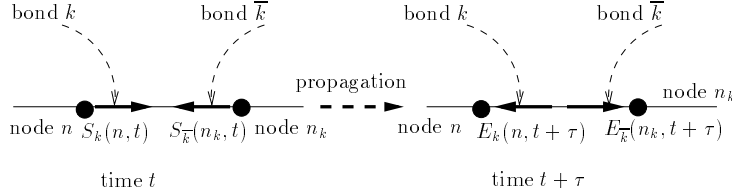


Fig. 6. Notations for the bond linking the central node n to one of its neighbor node n_k and the propagating currents defined on this bond.

ψ_R , satisfy the following discrete propagation equations

$$\begin{aligned} \psi_L(n, t + \tau) &= f_L(\psi_L(n', t'), \psi_R(n', t'), \psi_L(n'', t''), \psi_R(n'', t''), \dots), \end{aligned} \quad (8)$$

$$\begin{aligned} \psi_R(n, t + \tau) &= f_R(\psi_R(n', t'), \psi_L(n', t'), \psi_R(n'', t''), \psi_L(n'', t''), \dots), \end{aligned} \quad (9)$$

where the fields, ψ_L , ψ_R , on node n and at time $t + \tau$ are functions of both chiral fields on the same node and/or on nodes n' , n'' , \dots , at previous times t' , t'' , \dots . Equations (8, 9) are closed equations in the fields. This means that $f_{L,R}$ are functions involving the two chiral fields only and do not depend explicitly on the incident currents of both kinds. Moreover, the number of terms in the right hand side of equations (8, 9) must be finite. Those two closure conditions strongly constrain the form of the scattering matrices $S_{L,R}$, so that one finds

$$S_L = P_L + \text{diag}(\mu_{1,L}, \mu_{2,L}, \mu_{3,L}), \quad (10)$$

$$S_R = P_R + \text{diag}(\mu_{1,R}, \mu_{2,R}, \mu_{3,R}), \quad (11)$$

where

$$(P)_{ij,\alpha} = \rho_{i,\alpha} \lambda_{j,\alpha}, \quad i, j = 1, 2, 3, \quad \alpha = R, L, \quad (12)$$

and $\lambda_{j,\alpha}$ denotes the complex numbers appearing in the definitions of the chiral fields. The $\rho_{i,\alpha}$ and $\mu_{i,\alpha}$ are complex numbers to be determined. As usual diag denotes a diagonal matrix. The discussion leading to the form (10, 11) of the scattering matrices is exactly the same as for scalar fields [6]. Therefore, we have only quoted the result here. According to equations (10, 11), it is easy to check that the outgoing current S_k for each species splits into a term proportional to the incident current E_k of the same

kind on the same bond k and into a term proportional to the field of the same kind

$$S_{k,L} = \rho_{k,L} \psi_L + \mu_{k,L} E_{k,L}, \quad (13)$$

$$S_{k,R} = \rho_{k,R} \psi_R + \mu_{k,R} E_{k,R}, \quad k = 1, 2, 3. \quad (14)$$

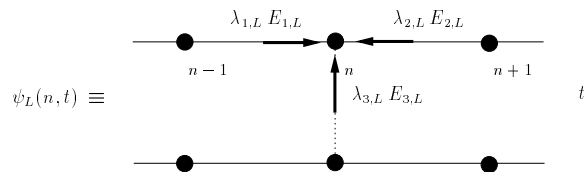
As the constructions of the equations (8, 9) obeyed by ψ_L and ψ_R are carried out in the same way, we only consider the equation for ψ_L in the following. An analogous equation can be derived for the field ψ_R . We fix the space-time units, a and τ , to unity. Let us consider the L field on node n at time $t + 1$ which is defined as (Eq. (6))

$$\psi_L(n, t + 1) = \sum_{k=1}^3 \lambda_{k,L} E_{k,L}(n, t + 1).$$

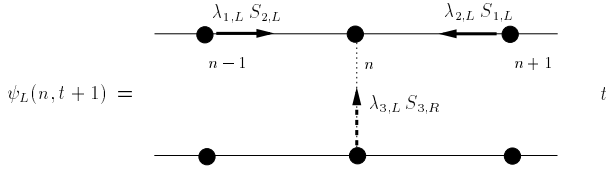
According to the propagation step, it can be expressed in terms of the outgoing currents at the previous time t

$$\psi_L(n, t + 1) = \sum_{k=1,2} \lambda_{k,L} S_{\bar{k},L}(n_k, t) + \lambda_{3,L} S_{3,R}(n, t). \quad (15)$$

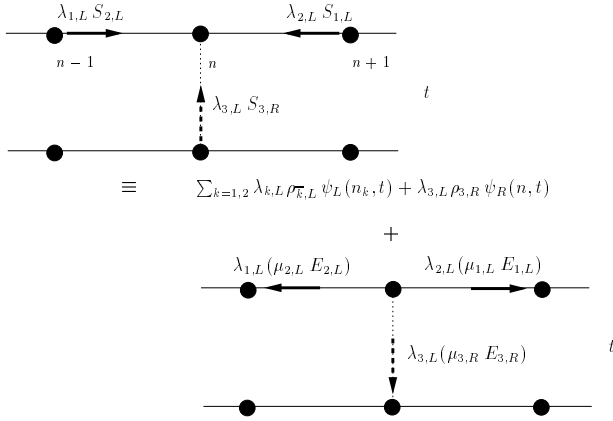
In the remainder of this derivation, we will use drawings as often as possible since those representations are far more transparent than formal equations. For example, the definition of the L field (Eq. (6)), $\psi_L(n, t) = \sum_{k=1}^3 \lambda_{k,L} E_{k,L}(n, t)$, is represented schematically as sketched below



whereas equation (15) is sketched



At time t , the outgoing bond currents appearing in equation (15) were instantaneously scattered by S_L matrices. Hence, $S_{1,L}$ and $S_{2,L}$ are linear superpositions of L incident currents on nodes $n+1$ and $n-1$ respectively, whereas $S_{3,R}$ is scattered by an S_R matrix on node n . The resulting linear superpositions are given by the specific form (13). Schematically, this scattering process is represented as follows



which reads analytically

$$\begin{aligned} \psi_L(n, t+1) &= \sum_{k=1,2} \lambda_{k,L} \rho_{k,L} \psi_L(n_k, t) \\ &\quad + \lambda_{3,L} \rho_{3,R} \psi_R(n, t) \\ &+ \underbrace{\sum_{k=1,2} \lambda_{k,L} \mu_{k,L} E_{k,L}(n_k, t) + \lambda_{3,L} \mu_{3,R} E_{3,R}(n, t)}_{\text{current term}}. \end{aligned} \quad (16)$$

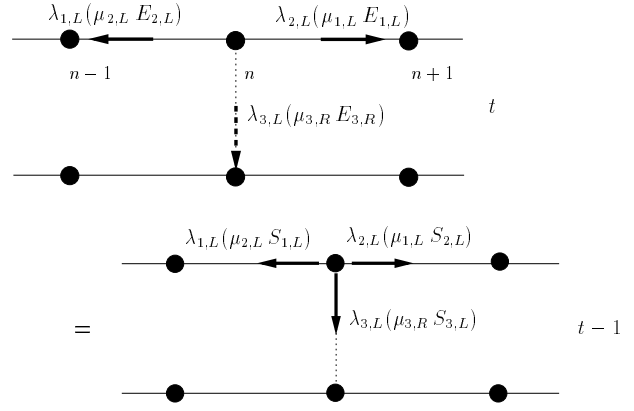


Fig. 7. The upper drawing is a sketch of the current term in equation (16) at time t , while the lower drawing is a sketch of the same currents at time $t-1$ before the propagation step.

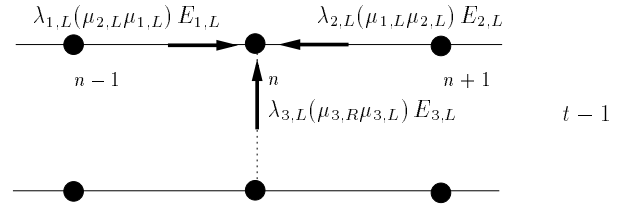


Fig. 8. Sketch of the new current term in equation (18).

According to the propagation step, the current term of equation (16), which is the sum involving only the incident currents and not the chiral fields, is deduced from three outgoing currents at time $t-1$. Those outgoing currents are all defined on node n , as sketched in Figure 7 which translates analytically through the following equation

$$\begin{aligned} \text{current term} &= \sum_{k=1,2} \lambda_{k,L} \mu_{k,L} S_{k,L}(n, t-1) \\ &\quad + \lambda_{3,L} \mu_{3,R} S_{3,L}(n, t-1). \end{aligned} \quad (17)$$

Again, the outgoing currents appearing in the right hand side of equation (17) are deduced from incident currents on the same node according to the scattering process (13). This leads to the equation (18).

$$\begin{aligned} \text{current term} &= \left(\sum_{k=1,2} \lambda_{k,L} \rho_{k,L} \mu_{k,L} + \lambda_{3,L} \rho_{3,R} \mu_{3,R} \right) \psi_L(n, t-1) \\ &\quad + \underbrace{\sum_{k=1,2} \lambda_{k,L} \mu_{k,L} \mu_{k,L} E_{k,L}(n, t-1) + \lambda_{3,L} \mu_{3,R} \mu_{3,L} E_{3,L}(n, t-1)}_{\text{new current term.}} \end{aligned} \quad (18)$$

The new current term in the last equation is schematically depicted in Figure 8.

Plugging the right hand side of equation (18) into equation (16) leads to

$$\begin{aligned} & \psi_L(n, t + 1) \\ & - \left(\sum_{k=1,2} \lambda_{k,L} \rho_{k,L} \mu_{\bar{k},L} + \lambda_{3,L} \rho_{3,L} \mu_{3,R} \right) \psi_L(n, t - 1) = \\ & \sum_{k=1,2} \lambda_{k,L} \rho_{\bar{k},L} \psi_L(n_k, t) + \lambda_{3,L} \rho_{3,R} \psi_R(n, t) \\ & + \sum_{k=1,2} (\mu_{\bar{k},L} \mu_{k,L}) \lambda_{k,L} E_{k,L}(n, t - 1) \\ & + (\mu_{3,R} \mu_{3,L}) \lambda_{3,L} E_{3,L}(n, t - 1). \end{aligned} \quad (19)$$

Equation (19) is almost in the closed form we are looking for but still contains a current term. In order to get an equation which involves solely L, R fields without any L, R currents, it is sufficient to impose the following relations

$$\begin{cases} \mu_{k,L} \mu_{\bar{k},L} = \mu_L^2, & k = 1, 2, \\ \mu_{3,R} \mu_{3,L} = \mu_L^2, \end{cases} \quad (20)$$

where μ_L is a constant parameter. The relations (20) have been chosen in such a way that the last term of equation (19) becomes proportional to $\psi_L(n, t - 1)$. The insertion of equations (20) into (19) leads to a discrete propagation equation of the type of equation (8):

$$\begin{aligned} & \psi_L(n, t + 1) - \mu_L^2 \left(1 + \sum_{k=1}^3 \frac{\lambda_{k,L} \rho_{k,L}}{\mu_{k,L}} \right) \psi_L(n, t - 1) \\ & = \sum_{k=1,2} \lambda_{k,L} \rho_{\bar{k},L} \psi_L(n_k, t) + \lambda_{3,L} \rho_{3,R} \psi_R(n, t). \end{aligned} \quad (21)$$

The discrete propagation equation obeyed by the R field is directly deduced from the equation for the L field by exchanging all the subscripts L and R :

$$\begin{aligned} & \psi_R(n, t + 1) - \mu_R^2 \left(1 + \sum_{k=1}^3 \frac{\lambda_{k,R} \rho_{k,R}}{\mu_{k,R}} \right) \psi_R(n, t - 1) \\ & = \sum_{k=1,2} \lambda_{k,R} \rho_{\bar{k},R} \psi_R(n_k, t) + \lambda_{3,R} \rho_{3,L} \psi_L(n, t), \end{aligned} \quad (22)$$

where equation (22) has been supplemented by sufficient conditions of the type of equations (20)

$$\begin{cases} \mu_{k,R} \mu_{\bar{k},R} = \mu_R^2, & k = 1, 2, \\ \mu_{3,L} \mu_{3,R} = \mu_R^2. \end{cases} \quad (23)$$

The comparison of the second equations in both systems (20, 23) leads to the equality: $\mu_L^2 = \mu_R^2$. In the rest of this section, we will denote this constant by μ^2 . Now, if the matrix parameters appearing in equations (21, 22) are

chosen according to the following relations

$$\mu^2 \left(1 + \sum_{k=1}^3 \frac{\lambda_{k,L} \rho_{k,L}}{\mu_{k,L}} \right) = \mu^2 \left(1 + \sum_{k=1}^3 \frac{\lambda_{k,R} \rho_{k,R}}{\mu_{k,R}} \right) = 1, \quad (24)$$

then a first-order discretized time derivative appears

$$\begin{aligned} \psi_L(n, t + 1) - \psi_L(n, t - 1) & = \sum_{k=1,2} \lambda_{k,L} \rho_{\bar{k},L} \psi_L(n_k, t) \\ & + \lambda_{3,L} \rho_{3,R} \psi_R(n, t), \end{aligned} \quad (25)$$

$$\begin{aligned} \psi_R(n, t + 1) - \psi_R(n, t - 1) & = \sum_{k=1,2} \lambda_{k,R} \rho_{\bar{k},R} \psi_R(n_k, t) \\ & + \lambda_{3,R} \rho_{3,L} \psi_L(n, t). \end{aligned} \quad (26)$$

On the same ground, the sums

$$\sum_{k=1,2} \lambda_{k,L,R} \rho_{\bar{k},L,R} \psi_{L,R}(n_k, t)$$

entering equations (25, 26) can be turned into first-order discretized spatial derivatives by an appropriate choice of the products $\lambda_{k,L,R} \rho_{\bar{k},L,R}$, for $k = 1, 2$. With this assumption, the two coupled discretized wave propagation equations (25, 26) can be identified with the Dirac equation in the Majorana-Weyl representation. However, we don't need this last assumption at this step of our derivation. In the next section we show that implementing some suitable symmetries on the scattering process constrains even more the L, R matrix coefficients and turns the terms $\sum_{k=1,2} \lambda_{k,L,R} \rho_{\bar{k},L,R} \psi_{L,R}(n_k, t)$ into first-order discretized spatial derivatives.

2.3 Symmetries of the scattering process

Our goal is now to prove that a few well chosen assumptions on the symmetries of the scattering process leads to the unique determination of the L, R scattering matrices and to the discrete propagation equations (25, 26) in a form very close to the Dirac equation (1). The symmetries are implemented both on the currents and on the fields. In this model, the scattering process is described by matrices which are unitary due to time-reversal invariance and reciprocity. This property insures the local conservation, at each time step, of the "flux" of the L, R currents: $(|S_1|^2 + |S_2|^2 + |S_3|^2)_{L,R} = (|E_1|^2 + |E_2|^2 + |E_3|^2)_{L,R}$, which is a stability condition on the dynamics of the L, R fields at large times. Finally, the last symmetry required to complete the construction is reflection symmetry ($x \mapsto -x$) which is known to connect the two chiral fields.

2.3.1 Time-reversal invariance

Time-reversal invariance for the currents can be formulated by stating that the two scattering processes, sketched in Figure 9, are equivalent under a time-reversal transformation.

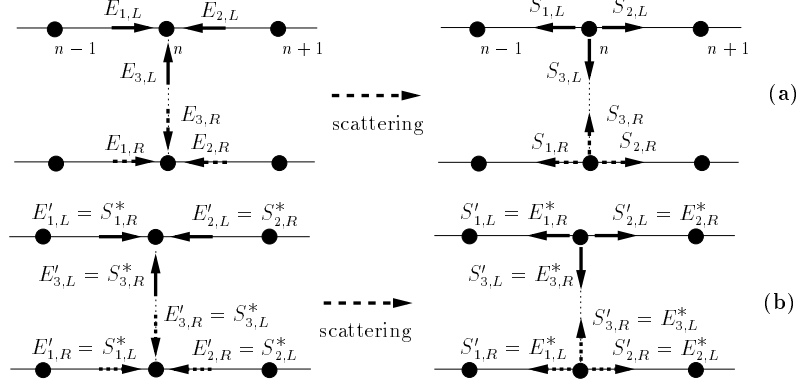


Fig. 9. Time-reversal invariance of the scattering process. The scattering process for the L, R currents depicted in (a) is equivalent to the scattering process deduced from time-reversal symmetry (b).

Note that the L and R currents are exchanged in this transformation. This corresponds to the exchange of the spinor components ψ_L and ψ_R when considering the time-reversal transformation in the Dirac equation. This leads to the following relation between the scattering matrices

$$S_L = (S_R^*)^{-1}. \quad (27)$$

According to the form (10, 11, 12) of the L, R scattering matrices, equation (27) leads to the following constraints on the L, R matrix coefficients

$$\begin{cases} \mu_{k,L} \frac{\rho_{k,R}^*}{\rho_{k,L}} = C, & \mu_{k,R} \frac{\lambda_{k,L}^*}{\lambda_{k,R}} = C', \\ \mu_{k,L} \frac{\lambda_{k,R}^*}{\lambda_{k,L}} = \frac{1}{C'^*}, & \mu_{k,R} \frac{\rho_{k,L}^*}{\rho_{k,R}} = \frac{1}{C^*}, \end{cases} \quad \text{for } k = 1, 2, 3, \quad (28)$$

where C, C' are two constants which obey the condition $C + C'^* = \sum_{k=1}^3 \lambda_{k,L} \rho_{k,R}^*$. Equations (28) allow to express the R matrix coefficients in terms of those of the L matrix.

We must also consider time-reversal invariance for the chiral fields ψ_L and ψ_R . For its implementation, it is convenient to introduce the scalar field ψ^S constructed as a linear superposition of outgoing currents

$$\psi^S(n, t) \equiv \sum_{k=1}^3 \kappa_k S_k(n, t), \quad (29)$$

where the coefficients κ_k are complex numbers. Up to now the scalar fields ψ_L^E, ψ_R^E , considered so far were linear superpositions of incident currents (Eqs. (6, 7)). Owing to the linearity of the scattering process (2, 3) there always exist numbers κ_k such that the fields ψ^S obey the same propagation equations as those satisfied by ψ^E , namely equations (25, 26). By taking into account the special form (13) of the outgoing currents, equation (29) for the L field becomes

$$\begin{aligned} \psi_L^S(n, t) &= \left(\sum_{k=1}^3 \kappa_{k,L} \rho_{k,L} \right) \psi_L^E(n, t) \\ &+ \sum_{k=1}^3 \kappa_{k,L} \mu_{k,L} E_{k,L}(n, t). \end{aligned}$$

It is easy to show that the condition $\kappa_{k,L} \mu_{k,L} = \kappa_L \lambda_{k,L}$, where κ_L doesn't depend on the subscript k and is given by $\kappa_L^{-1} = 1 + \sum_{k=1}^3 \lambda_{k,L} \rho_{k,L} / \mu_{k,L}$, leads to the equality between the outgoing field and the incident field: $\psi_L^S = \psi_L^E$. Finally, using equations (24), ψ_L^S reads

$$\psi_L^S(n, t) \equiv \sum_{k=1}^3 \kappa_{k,L} S_{k,L}(n, t) = \mu^2 \sum_{k=1}^3 \frac{\lambda_{k,L}}{\mu_{k,L}} S_{k,L}(n, t). \quad (30)$$

Similarly, one finds

$$\psi_R^S(n, t) \equiv \sum_{k=1}^3 \kappa_{k,R} S_{k,R}(n, t) = \mu^2 \sum_{k=1}^3 \frac{\lambda_{k,R}}{\mu_{k,R}} S_{k,R}(n, t). \quad (31)$$

It is natural to implement time-reversal invariance on the chiral fields by the following conditions

$$\psi_L^S(n, t) = \psi_R^{E*}(n, t), \quad (32)$$

$$\psi_R^S(n, t) = \psi_L^{E*}(n, t). \quad (33)$$

Here, we have explicitly distinguished the fields constructed on the outgoing currents, $\psi_{L,R}^S$, and the fields constructed on the incident currents, $\psi_{L,R}^E$. Moreover the complex conjugate fields have been considered in the inverse process. Now if we write down the currents in equations (32, 33) according to the definitions of the fields (6, 7, 29) the time-reversal invariance of the scattering process (Fig. 9) leads to the following relations

$$\kappa_{k,L} = \lambda_{k,R}^*, \quad \kappa_{k,R} = \lambda_{k,L}^* \quad \text{for } k = 1, 2, 3. \quad (34)$$

Then, using equations (20, 23) with $\mu_L^2 = \mu_R^2 = \mu^2$, the last equation (34) gives

$$\begin{cases} \mu_{k,R} \mu_{k,L}^* = 1, & k = 1, 2, 3, \text{ and } |\mu| = 1, \\ \mu_{\bar{k},L} = \mu^2 \mu_{k,R}^*, & k = 1, 2. \end{cases} \quad (35)$$

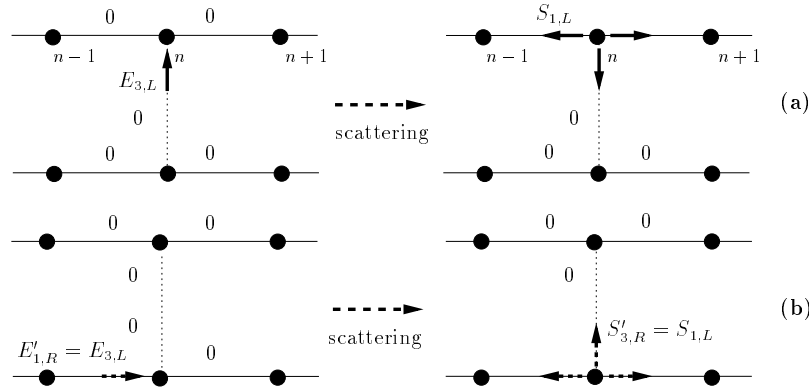


Fig. 10. Reciprocity principle of the scattering process. The scattering channel $3 \rightarrow 1$ for the L currents (a) is equivalent to the scattering channel $1 \rightarrow 3$ for the R currents (b).

Equations (28, 34, 35) lead to: $C' = \mu^2$. Lastly, the relations (28) read

$$\begin{cases} \mu_{k,L} \frac{\lambda_{k,R}^*}{\lambda_{k,L}} = \mu^2, & \mu_{k,R} \frac{\lambda_{k,L}^*}{\lambda_{k,R}} = \mu^2, \\ \mu_{k,L} \frac{\rho_{k,R}^*}{\rho_{k,L}} = C, & \mu_{k,R} \frac{\rho_{k,L}^*}{\rho_{k,R}} = \frac{1}{C^*}, \end{cases} \quad \text{for } k = 1, 2, 3, \quad (36)$$

with $C + (\mu^*)^2 = \sum_{k=1}^3 \lambda_{k,L} \rho_{k,R}^*$.

2.3.2 Reciprocity principle

The reciprocity principle states that the response of a system at a point \mathbf{r}' due to an excitation at point \mathbf{r} is identical to the reciprocal process, namely the response of the system at point \mathbf{r} due to an excitation at point \mathbf{r}' . In our current model, the application of this principle to the scattering process means that any scattering channel is identical to its reciprocal channel. The definition of a scattering channel is illustrated in Figure 10a. An incident L current at node n , $E_{3,L}$ in Figure 10a, is scattered in three outgoing currents $S_{i,L}$. A scattering channel is any of the three elementary processes, $E_{3,L} \rightarrow S_{1,L}$, $E_{3,L} \rightarrow S_{2,L}$ and $E_{3,L} \rightarrow S_{3,L}$. Let us consider the first channel $E_{3,L} \rightarrow S_{1,L}$. The reciprocity principle states that this scattering channel is equivalent to the reciprocal channel for the R current illustrated in Figure 10b. Roughly speaking, each scattering channel for L currents in one direction is equivalent to the same scattering channel for R currents in the opposite direction. As for the time-reversed processes considered in the previous section, note that the L and R currents are exchanged in the two reciprocal processes. This exchange of L and R currents is suggested by the exchange of the spinor components ψ_L and ψ_R in the symmetries underlying the Dirac equation.

Finally, reciprocity as defined above leads to the following relation between the scattering matrices

$$S_L = {}^T S_R, \quad (37)$$

and to the following constraints on the L, R matrix coefficients

$$\begin{cases} \rho_{k,L} = \gamma \lambda_{k,R}, \\ \rho_{k,R} = \gamma \lambda_{k,L}, \\ \mu_{k,L} = \mu_{k,R}, \end{cases} \quad \text{for } k = 1, 2, 3, \quad (38)$$

where γ is a complex number.

2.3.3 Unitarity of the scattering process

The combination of equations (27, 37) shows that S_L and S_R are unitary matrices, each one satisfying:

$$SS^\dagger = Id, \quad (39)$$

where Id denotes the identity matrix. Unitarity which is expressed by combining together equations (35, 36, 38) leads to

$$\begin{cases} |\mu_{k,L}| = |\mu_{k,R}| = 1, \\ \mu_{k,L} \frac{\lambda_{k,L}^*}{\rho_{k,L}} = \frac{C}{\gamma^*}, \\ \mu_{k,R} \frac{\lambda_{k,R}^*}{\rho_{k,R}} = \frac{1}{(C\gamma)^*}, \end{cases} \quad \text{for } k = 1, 2, 3, \quad (40)$$

where $C/\gamma + (C/\gamma)^* = -\Lambda$, with $\Lambda \equiv \sum_{k=1,2,3} |\lambda_{k,L}|^2 = \sum_{k=1,2,3} |\lambda_{k,R}|^2$. Finally, some manipulations using equations (24, 38, 40) yields the parametrization of the L, R matrix coefficients in terms of $\lambda_{k,L}$, $\mu_{k,L}$ and μ

$$\begin{cases} \rho_{k,L} = \frac{\mu^* - \mu}{\Lambda} \left(\frac{\mu_{k,L}}{\mu} \right) \lambda_{k,L}^*, \\ \lambda_{k,R} = \frac{1}{\mu} \left(\frac{\mu_{k,L}}{\mu} \right) \lambda_{k,L}^*, \\ \rho_{k,R} = \frac{\mu^* - \mu}{\Lambda} \mu \lambda_{k,L}. \end{cases} \quad \text{for } k = 1, 2, 3, \quad (41)$$

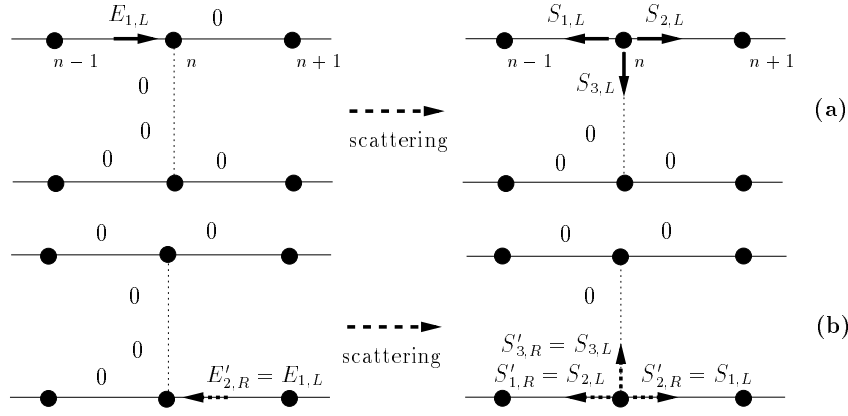


Fig. 11. Reflection invariance of the scattering process for the left bond current. The scattering process for the L current $E_{1,L}$ (a) is equivalent to the scattering process after a reflection symmetry for the R current $E'_{2,R}$ (b).

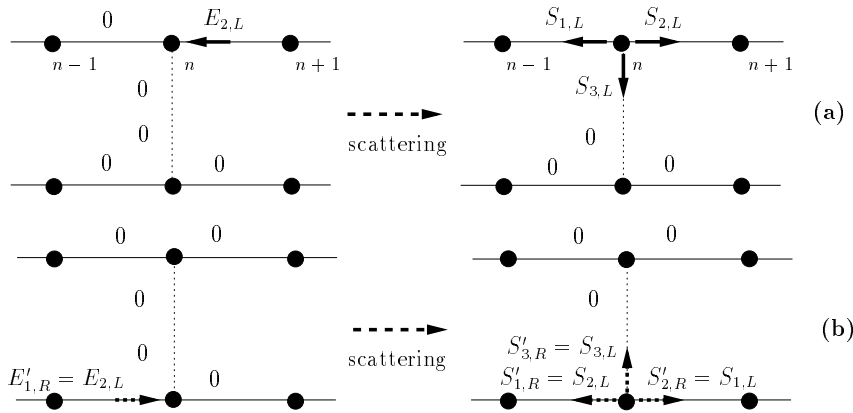


Fig. 12. Reflection invariance of the scattering process for the right bond current. The scattering process for the L current $E_{2,L}$ (a) is equivalent to the scattering process after a reflection symmetry for the R current $E'_{1,R}$ (b).

Using this parametrization and choosing $\mu_{3,L} = \mu_{3,R} = \mu$ from equations (20, 38) the scattering matrices S_L and S_R , (Eqs. (10, 11, 12)) take the form

$$s_{kl,L} = \frac{\mu^* - \mu}{\Lambda} \left(\frac{\mu_{k,L}}{\mu} \right) \lambda_{k,L}^* \lambda_{l,L} + \mu_{k,L} \delta_{kl}, \quad (42)$$

$$s_{ij,R} = \frac{\mu^* - \mu}{\Lambda} \left(\frac{\mu_{j,L}}{\mu} \right) \lambda_{i,L} \lambda_{j,L}^* + \mu_{i,L} \delta_{ij}, \quad (43)$$

$$\begin{aligned} \psi_R(n, t+1) - \psi_R(n, t-1) = & \frac{\mu^* - \mu}{\Lambda} \left[\left(\frac{\mu_{2,L}}{\mu} \right) \lambda_{1,L} \lambda_{2,L}^* \psi_R(n+1, t) \right. \\ & \left. + \left(\frac{\mu_{1,L}}{\mu} \right) \lambda_{1,L}^* \lambda_{2,L} \psi_R(n-1, t) \right] \\ & + \left(\frac{\mu^* - \mu}{\Lambda} \right)^2 \frac{\lambda_{3,L}^*{}^2}{\gamma} \psi_L(n, t). \end{aligned} \quad (45)$$

and the wave propagation equations (25, 26) become

$$\begin{aligned} \psi_L(n, t+1) - \psi_L(n, t-1) = & \frac{\mu^* - \mu}{\Lambda} \left[\left(\frac{\mu_{1,L}}{\mu} \right) \lambda_{1,L}^* \lambda_{2,L} \psi_L(n+1, t) \right. \\ & \left. + \left(\frac{\mu_{2,L}}{\mu} \right) \lambda_{1,L} \lambda_{2,L}^* \psi_L(n-1, t) \right] + \gamma \lambda_{3,L}^2 \psi_R(n, t), \end{aligned} \quad (44)$$

2.3.4 Reflection invariance

Reflection invariance states that the Dirac equation (1) is invariant by the simultaneous exchange of ψ_L into ψ_R and x into $-x$. We transpose this property to the scattering process of the currents. Reflection invariance of the scattering process can be decomposed in three elementary steps sketched in Figures 11, 12 and 13. This symmetry is formulated by stating that the two scattering processes represented in figures (a) and (b) respectively, are equivalent under a reflection.

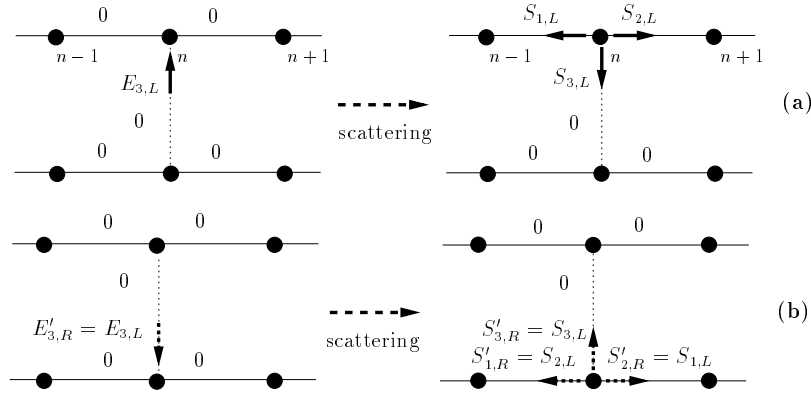


Fig. 13. Reflection invariance of the scattering process for the node current. The scattering process for $E_{3,L}$ (a) is equivalent to the scattering process after a reflection symmetry for $E'_{3,R}$ (b).

$$\begin{aligned} \psi_L(n, t+1) - \psi_L(n, t-1) = \\ \varepsilon_\mu \frac{(\mu^* - \mu)}{2 + |\nu|^2} \frac{\lambda_{2,L}}{\lambda_{1,L}} \left[\psi_L(n+1, t) + \left(\frac{\lambda_{2,L}}{\lambda_{1,L}} \right)^* \frac{\lambda_{1,L}}{\lambda_{2,L}} \psi_L(n-1, t) \right] + \varepsilon_\gamma \left(\frac{(\mu^* - \mu)|\nu|^2}{2 + |\nu|^2} \right) \psi_R(n, t), \end{aligned} \quad (52)$$

$$\begin{aligned} \psi_R(n, t+1) - \psi_R(n, t-1) = \\ \varepsilon_\mu \frac{(\mu^* - \mu)}{2 + |\nu|^2} \left(\frac{\lambda_{2,L}}{\lambda_{1,L}} \right)^* \left[\psi_R(n+1, t) + \frac{\lambda_{2,L}}{\lambda_{1,L}} \left(\frac{\lambda_{1,L}}{\lambda_{2,L}} \right)^* \psi_R(n-1, t) \right] + \varepsilon_\gamma \left(\frac{(\mu^* - \mu)|\nu|^2}{2 + |\nu|^2} \right) \psi_L(n, t). \end{aligned} \quad (53)$$

For example, the equivalence of the scattering processes (a) and (b) depicted in Figure 11 leads to the following relations between the matrix elements: $s_{kl,L} = s_{\bar{k}\bar{l},R}$, $k = 1, 2, 3$, and $l = 1$. Similar relations between $s_{kl,L}$ and $s_{\bar{k}\bar{l},R}$ hold from Figures 12 and 13. Finally, the constraints imposed by the reflection invariance on the scattering coefficients are summed up by the equation

$$s_{kl,L} = s_{\bar{k}\bar{l},R}, \quad \text{for } k, l = 1, 2, 3. \quad (46)$$

Taking into account the form of the L and R matrix elements (Eqs. (42, 43)), we deduce from (46)

$$\left\{ \begin{array}{l} \mu_{1,L} = \mu_{2,L}, \\ \frac{|\lambda_{2,L}|}{|\lambda_{1,L}|} = 1, \\ \left(\frac{\lambda_{3,L}}{\lambda_{1,L}} \right)^* \frac{\lambda_{1,L}}{\lambda_{3,L}} = \frac{\mu_{1,L}}{\mu} \left(\frac{\lambda_{2,L}}{\lambda_{1,L}} \right)^*, \end{array} \right. \quad (47)$$

where, using (35) and the third equation of system (38), $\mu_{1,L}$ and $\mu_{2,L}$ can be written

$$\mu_{1,L} = \mu_{2,L} = \varepsilon_\mu \mu, \quad \varepsilon_\mu = \pm 1. \quad (48)$$

An additional constraint arises by requiring the reflection invariance of the propagation equations (44, 45) satisfied by the chiral fields. If we plug the last result (48) in those two equations, the L, R fields are parity invariant provided the last two terms of the r.h.s. of equations (44, 45) are

equal, which implies

$$\gamma = \varepsilon_\gamma \left(\frac{\mu^* - \mu}{\Lambda_L} \right) \left(\frac{\lambda_{3,L}^*}{\lambda_{3,L}} \right), \quad \varepsilon_\gamma = \pm 1. \quad (49)$$

Using equations (47, 49), we obtain the phase of $\lambda_{3,L}$

$$\frac{\lambda_{3,L}^*}{\lambda_{3,L}} = \varepsilon_\gamma \mu. \quad (50)$$

2.4 Identification with the Dirac equation and exact form of the scattering matrices

In this section, we show that the propagation equations (44, 45) take either the form of the Dirac equation in the Majorana-Weyl representation or can be identified with two coupled Schrödinger equations (see Appendix). We first note that the propagation equations (44, 45) depend only on the ratios

$$\frac{\lambda_{2,L}}{\lambda_{1,L}}, \quad \text{and} \quad \nu \equiv \frac{\lambda_{3,L}}{\lambda_{1,L}}. \quad (51)$$

This is not surprising since the problem is linear which supposes that the chiral fields are defined up to a multiplicative constant: $\lambda_{1,L}$. Furthermore, according to the equality between the modulus of $\lambda_{2,L}$ and $\lambda_{1,L}$ (Eq. (47)) and to the definition of Λ , those two equations can be recast in the form

see equations (52) and (53) above.

If we restrict the ratio $(\lambda_{2,L}/\lambda_{1,L})^*/(\lambda_{2,L}/\lambda_{1,L})$ to be equal to ± 1 , *i.e.* $\lambda_{2,L}/\lambda_{1,L}$ is either a real number or a purely imaginary number, then the r.h.s. of the propagation equations (52, 53) displays a second order discretized spatial derivative or a first-order discretized spatial derivative, respectively. We postpone the case where $\lambda_{2,L}/\lambda_{1,L}$ is a real number to the appendix and concentrate on the other alternative where $\lambda_{2,L}/\lambda_{1,L}$ is a purely imaginary number. Combining $(\lambda_{2,L}/\lambda_{1,L})^* = -(\lambda_{2,L}/\lambda_{1,L})$ with $|\lambda_{2,L}/\lambda_{1,L}| = 1$ (Eq. (47)) leads to $\lambda_{2,L}/\lambda_{1,L} = i\varepsilon_\lambda$, where $\varepsilon_\lambda = \pm 1$. Writing $\mu = \exp(i\theta)$ (see Eq. (35)), the propagation equations (52, 53) depend only on the two signs $\varepsilon \equiv \varepsilon_\lambda \varepsilon_\mu$, ε_γ , and on the two real parameters $|\nu|$, θ . The space-time units, a and τ , are again introduced so that $c_0 = a/\tau$ is a microscopic velocity. The choices $\varepsilon = -1$ and $\varepsilon_\gamma = 1$ leads to the identification with the Dirac equation in the Majorana-Weyl representation (1)

$$\begin{aligned} & \frac{1}{2\tau} (\psi_L(n, t + \tau) - \psi_L(n, t - \tau)) = \\ & - \left(\frac{2 \sin \theta}{2 + |\nu|^2} \right) \frac{a}{\tau} \left(\frac{\psi_L(n + a, t) - \psi_L(n - a, t)}{2a} \right) \\ & - \frac{i}{\tau} \left(\frac{\sin \theta |\nu|^2}{2 + |\nu|^2} \right) \psi_R(n, t), \end{aligned} \quad (54)$$

$$\begin{aligned} & \frac{1}{2\tau} (\psi_R(n, t + \tau) - \psi_R(n, t - \tau)) = \\ & - \left(\frac{2 \sin \theta}{2 + |\nu|^2} \right) \frac{a}{\tau} \left(\frac{\psi_R(n + a, t) - \psi_R(n - a, t)}{2a} \right) \\ & - \frac{i}{\tau} \left(\frac{\sin \theta |\nu|^2}{2 + |\nu|^2} \right) \psi_L(n, t). \end{aligned} \quad (55)$$

Comparison between equations (54, 55) and (1) provides the velocity c and the mass m of the chiral fields

$$c = c_0 \frac{2 \sin \theta}{2 + |\nu|^2}, \quad (56)$$

$$\frac{mc^2}{\hbar} = \frac{1 \sin \theta |\nu|^2}{\tau (2 + |\nu|^2)}. \quad (57)$$

Exact form of the scattering matrices

The L, R matrix coefficients (42, 43) depend only on μ , on the ratios $\mu_{1,L}/\mu = \mu_{2,L}/\mu = \varepsilon_\mu$, $\lambda_{2,L}/\lambda_{1,L} = i\varepsilon_\lambda = -i\varepsilon_\mu$, and ν . However the phase of ν is fixed because $\nu^*/\nu = i$, thanks to the third equation of (47) and to (48). Therefore, the L, R scattering matrices are parametrized by three real parameters: ε_μ , θ and $|\nu|$. We introduce a complex number α defined by

$$\alpha^2 = i\varepsilon_\mu, \quad (58)$$

so that the scattering matrices read

$$S_L = -\frac{2 \sin \theta}{2 + |\nu|^2} \alpha^2 \begin{pmatrix} 1 & -\alpha^2 & \nu \\ \alpha^2 & 1 & \alpha^2 \nu \\ \alpha^2 \nu & \nu & \alpha^2 \nu^2 \end{pmatrix} + \mu \text{diag}(\varepsilon_\mu, \varepsilon_\mu, 1), \quad (59)$$

$$S_R = -\frac{2 \sin \theta}{2 + |\nu|^2} \alpha^2 \begin{pmatrix} 1 & \alpha^2 & \alpha^2 \nu \\ -\alpha^2 & 1 & \nu \\ \nu & \alpha^2 \nu & \alpha^2 \nu^2 \end{pmatrix} + \mu \text{diag}(\varepsilon_\mu, \varepsilon_\mu, 1). \quad (60)$$

Lastly, one finds easily the following relations

$$\frac{\lambda_{1,L}^*}{\lambda_{1,L}} = -i\mu, \quad \frac{\lambda_{2,L}^*}{\lambda_{2,L}} = i\mu, \quad (61)$$

and

$$\lambda_{1,R} = -i\varepsilon_\mu \lambda_{1,L}, \quad \lambda_{2,R} = i\varepsilon_\mu \lambda_{2,L}, \quad \lambda_{3,R} = \lambda_{3,L}, \quad (62)$$

which will be used later in the standard representation (Sect. 3.3).

3 Standard representation

In this section, the ideas developed in the construction of the current model for the Majorana-Weyl representation are reproduced for the standard representation in $1 + 1$ space-time dimensions. Let us recall the Hamiltonian form of the Dirac equation in the standard representation

$$i \frac{\partial \Psi}{\partial t} = \left[-ic\sigma_1 \frac{\partial}{\partial x} + \frac{mc^2}{\hbar} \sigma_3 \right] \Psi, \quad (63)$$

where $\Psi = (\psi_+ \ \psi_-)^T$ is the two component spinor field in the standard representation. Surprisingly, it turns out that the scattering matrices cannot be computed by a direct construction as in the Majorana-Weyl representation. More precisely, it appears that the natural symmetries of the Dirac equation in the standard representation transposed to the scattering process lead to contradictions in the computation of the scattering matrices. However, if we take advantage of its linear connection with the Majorana-Weyl representation, we are able to build indirectly the current model for the standard representation.

3.1 Basic definitions: currents and fields

The two current models constructed for the standard and the Majorana-Weyl representations share many features. Indeed, two kinds of incident and outgoing currents are defined on the bonds and on the nodes of a regular chain

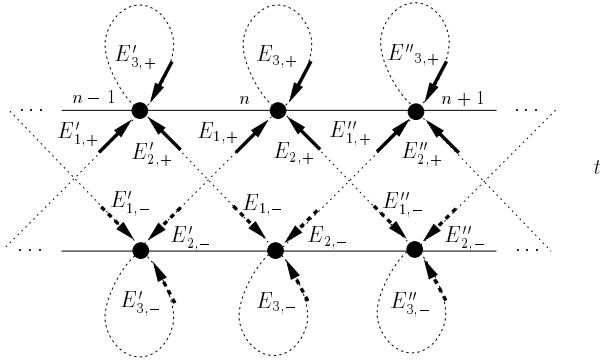


Fig. 14. Sketch of the chain in the standard representation. The upper and lower subchains carry the + and - currents respectively.

as the goal is to describe the propagation equations obeyed by the two scalar fields ψ_+ and ψ_- . The lattice mesh size a and the elementary time step τ are taken equal to unity. Transposing the notations of Section 2.1, incident currents are denoted by $E_{k,+}$, $E_{l,-}$, whereas outgoing currents are denoted by $S'_{k,+}$, $S'_{l,-}$. Three incident currents of each kind live on each node of the chain depicted in Figure 14. $E_{1,2,\pm}$ denotes the \pm bond-currents and $E_{3,\pm}$ denotes the node-currents. The chain schematically represented in Figure 14 is deduced from the chain depicted in Figure 3, for the Majorana-Weyl representation, by replacing the L currents by + currents and the R currents by - currents.

The dynamics of the \pm currents is similar to the dynamics defined for the L, R currents in the Majorana-Weyl representation. The \pm currents obey a discrete Huygens principle on the chain which decomposes in two steps: a propagation step and a scattering step. At a given time t , all the \pm currents undergo the dynamics simultaneously. The incident currents $E_{j,+}$, $E_{l,-}$, are scattered instantaneously on each node of the chain to become outgoing currents $S_{i,+}$, $S_{k,-}$, which are linear superposition of those incident currents

$$S_{i,+}(n, t) = \sum_{j=1}^3 s_{ij,+} E_{j,+}(n, t), \quad i = 1, 2, 3, \quad (64)$$

$$S_{k,-}(n, t) = \sum_{l=1}^3 s_{kl,-} E_{l,-}(n, t), \quad k = 1, 2, 3. \quad (65)$$

where $s_{ij,+}$, $s_{kl,-}$ are the complex elements of the scattering matrices S_{\pm} attached to each node of the chain. Then, in a unit time step, the \pm outgoing currents propagate from one node to the nearest-neighbor ones to become incident currents at time $t + 1$. Using the notations previously introduced in Figure 6, the propagation rules read

$$\begin{cases} E_{k,+}(n, t+1) = S_{\bar{k},-}(n_k, t), & k = 1, 2, \\ E_{3,+}(n, t+1) = S_{3,+}(n, t), \\ E_{l,-}(n, t+1) = S_{l,+}(n_l, t), & l = 1, 2, \\ E_{3,-}(n, t+1) = -S_{3,-}(n, t). \end{cases} \quad (66)$$

Contrary to the construction of the current-model in the Majorana-Weyl representation the transmutation affects solely the bond-currents and not the node-currents. The choice of such propagation rules stems from the structure of the Dirac equation (63) in which the spatial derivative $\sigma_1 \partial / \partial x$ couples ψ_+ and ψ_- while the last term $(mc^2 / \hbar) \sigma_3$ does not. Additionally, a minus sign affects the propagation of the - node-current. The reason for introducing this minus sign will be clarified at the end of the construction (see Sect. 3.2). The dynamics of the \pm currents is summarized in Figure 15.

Finally, similarly to the definitions (6, 7) of the chiral fields in the Majorana-Weyl representation, we suppose that the scalar fields ψ_+ and ψ_- are defined on each site of the chain as complex linear superpositions of + or - incident currents respectively

$$\psi_+(n, t) = \sum_{k=1}^3 \lambda_{k,+} E_{k,+}(n, t), \quad (67)$$

$$\psi_-(n, t) = \sum_{l=1}^3 \lambda_{l,-} E_{l,-}(n, t), \quad (68)$$

where $\lambda_{k,+}$, $\lambda_{l,-}$ are complex numbers.

3.2 Discrete propagation equations linking ψ_+ and ψ_-

Given the local rules which govern the dynamics of the \pm currents, our aim is to derive two closed propagation equations involving the fields ψ_{\pm}

$$\begin{aligned} \psi_+(n, t+1) &= f_+(\psi_+(n', t'), \psi_-(n', t'), \psi_+(n'', t''), \psi_-(n'', t''), \dots), \\ & \quad (69) \end{aligned}$$

$$\begin{aligned} \psi_-(n, t+1) &= f_-(\psi_-(n', t'), \psi_+(n', t'), \psi_-(n'', t''), \psi_+(n'', t''), \dots), \\ & \quad (70) \end{aligned}$$

where the fields ψ_{\pm} on node n and at time $t + 1$ are functions of both fields on the same node and/or on neighboring nodes n' , n'' , \dots , at previous times t' , t'' , \dots . The necessary conditions constraining the scattering matrices S_{\pm} for the equations (69, 70) to be closed remain the same as in the Majorana-Weyl construction (see Sect. 2.2). So that the scattering matrices are of the form

$$S_+ = P_+ + \text{diag}(\mu_{1,+}, \mu_{2,+}, \mu_{3,+}), \quad (71)$$

$$S_- = P_- + \text{diag}(\mu_{1,-}, \mu_{2,-}, \mu_{3,-}), \quad (72)$$

where P_{\pm} are matrices which are generically given, for each kind of current, by (12) (with $\alpha = \pm$). The above form of the scattering matrices S_+ and S_- implies for the outgoing currents the following equations

$$S_{k,+} = \rho_{k,+} \psi_+ + \mu_{k,+} E_{k,+}, \quad (73)$$

$$S_{k,-} = \rho_{k,-} \psi_- + \mu_{k,-} E_{k,-}, \quad k = 1, 2, 3. \quad (74)$$

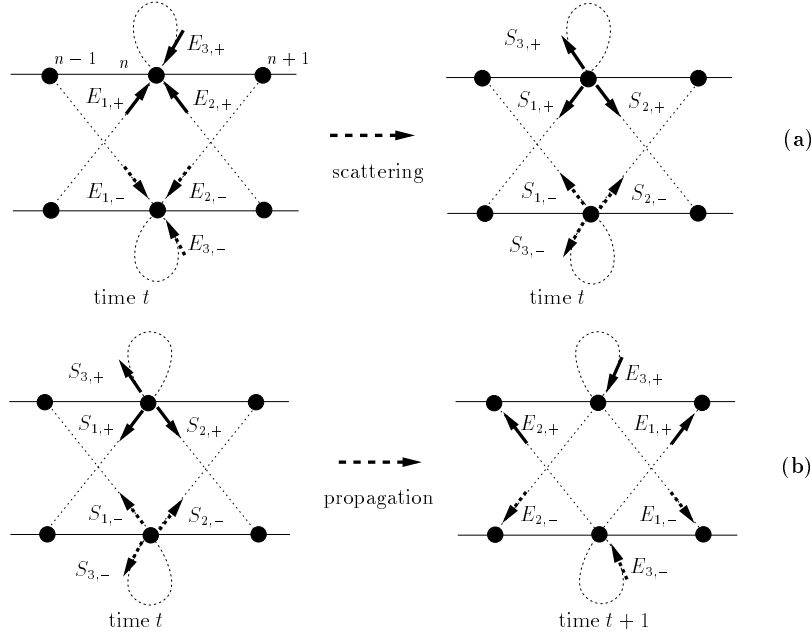


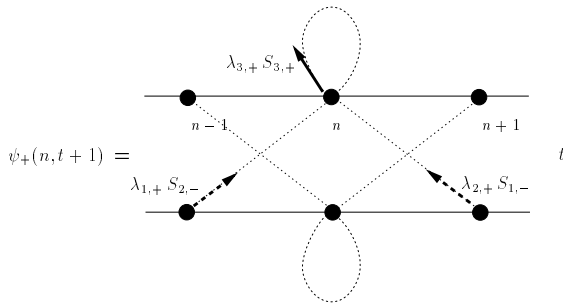
Fig. 15. Scattering step (a) and propagation step (b) for the + and - currents.

The rest of the derivation is almost the same as for the propagation equations (21, 22) in the Majorana-Weyl representation.

Using the propagation rules (66) and the definition (67), the field ψ_+ on node n at time $t + 1$ can be expressed in terms of \pm outgoing currents, defined at the previous time t and on the neighbor nodes n_k ,

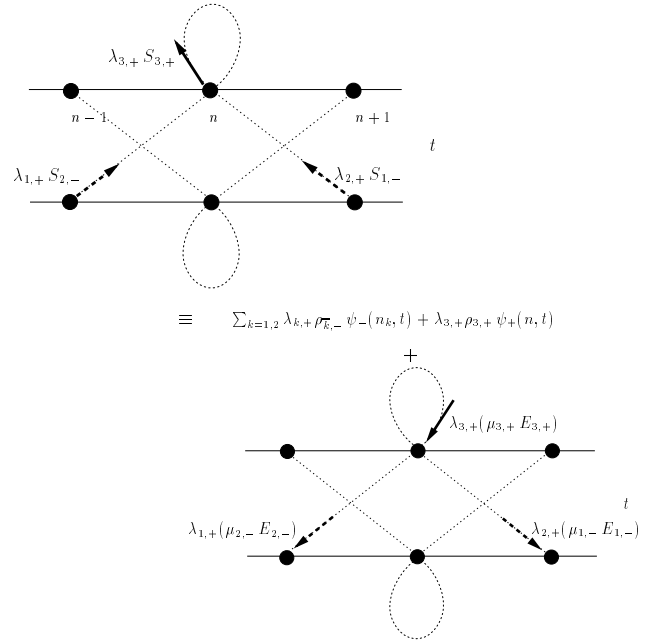
$$\psi_+(n, t+1) = \sum_{k=1,2} \lambda_{k,+} S_{k,-}^-(n_k, t) + \lambda_{3,+} S_{3,+}(n, t). \quad (75)$$

Equation (75) is sketched below



The outgoing currents entering equation (75) were scattered instantaneously at time t : $S_{1,-}$, $S_{2,-}$, and $S_{3,+}$, result from linear combinations of ingoing currents of the form (73, 74). As a consequence, $\psi_+(n, t+1)$ is represented schematically in terms of \pm ingoing currents at time t as

depicted as follows:



which reads analytically

$$\begin{aligned} & \psi_+(n, t+1) \\ &= \sum_{k=1,2} \lambda_{k,+} \rho_{k,-} \psi_-(n_k, t) + \lambda_{3,+} \rho_{3,+} \psi_+(n, t) \\ &+ \underbrace{\sum_{k=1,2} \lambda_{k,+} \mu_{k,-} E_{k,-}^-(n_k, t) + \lambda_{3,+} \mu_{3,+} E_{3,+}(n, t)}_{\text{current term}}. \quad (76) \end{aligned}$$

In virtue of the propagation and transmutation rules, the current term of equation (76) is deduced from three \pm outgoing currents that were defined on node n and at the previous time $t - 1$

current term

$$= \sum_{k=1,2} \lambda_{k,+} \mu_{\bar{k},-} S_{k,+}(n, t - 1) + \lambda_{3,+} \mu_{3,+} S_{3,+}(n, t - 1). \quad (77)$$

Applying the scattering rules to the outgoing currents entering the last equation gives rise to a field ψ_+ and to three $+$ incident currents, at time $t - 1$

current term

$$= \left(\sum_{k=1,2} \lambda_{k,+} \mu_{\bar{k},-} \rho_{k,+} + \lambda_{3,+} \mu_{3,+} \rho_{3,+} \right) \psi_+(n, t - 1) + \sum_{k=1,2} \lambda_{k,+} \mu_{\bar{k},-} \mu_{k,+} E_{k,+}(n, t - 1) + \lambda_{3,+} \mu_{3,+}^2 E_{3,+}(n, t - 1). \quad (78)$$

At this stage, we see that in order to close equation (76) at time $t - 1$, it is sufficient to impose

$$\begin{cases} \mu_{k,+} \mu_{\bar{k},-} = \mu_{+}^2, & k = 1, 2, \\ \mu_{3,+}^2 = \mu_{+}^2, \end{cases} \quad (79)$$

where we have introduced the constant μ_{+}^2 . We end up the derivation of the closed propagation equation satisfied by ψ_+ by plugging the current term (78) into equation (76), supplied with the two conditions (79),

$$\psi_+(n, t + 1) - \mu_{+}^2 \left(1 + \sum_{k=1}^3 \frac{\lambda_{k,+} \rho_{k,+}}{\mu_{k,+}} \right) \psi_+(n, t - 1) = \sum_{k=1,2} \lambda_{k,+} \rho_{\bar{k},-} \psi_-(n_k, t) + \lambda_{3,+} \rho_{3,+} \psi_+(n, t). \quad (80)$$

The propagation equation for ψ_- can be directly deduced from equation (80) by inverting all the subscripts $+$ and $-$, except that we must take care of the minus sign involved in the propagation step (66) of the $-$ node current. The propagation equation for ψ_- is obtained in its final form by noting that the minus sign appears first in the expression of $\psi_-(n, t + 1)$ involving \pm outgoing currents (equation analogous to Eq. (75))

$$\psi_-(n, t + 1) = \sum_{k=1,2} \lambda_{k,-} S_{\bar{k},+}(n_k, t) - \lambda_{3,-} S_{3,-}(n, t).$$

Then, according to the scattering step, the last term of this equation gives rise to the term: $-\lambda_{3,-} \rho_{3,-} \psi_-(n, t)$, which is analogous to the term $\lambda_{3,+} \rho_{3,+} \psi_+(n, t)$ in equation (80). Finally, one can check that this term is the only one affected by the minus sign so that the propagation

equation for ψ_- reads

$$\psi_-(n, t + 1) - \mu_{-}^2 \left(1 + \sum_{k=1}^3 \frac{\lambda_{k,-} \rho_{k,-}}{\mu_{k,-}} \right) \psi_-(n, t - 1) = \sum_{k=1,2} \lambda_{k,-} \rho_{\bar{k},+} \psi_+(n_k, t) - \lambda_{3,-} \rho_{3,-} \psi_-(n, t). \quad (81)$$

Equation (81) is supplied with the following sufficient conditions

$$\begin{cases} \mu_{k,-} \mu_{\bar{k},+} = \mu_{-}^2, & k = 1, 2, \\ \mu_{3,-}^2 = \mu_{-}^2. \end{cases} \quad (82)$$

Comparison between equations (79) and (82) shows that $\mu_{+}^2 = \mu_{-}^2$ which will be noted μ^2 in the following.

3.3 Transformation equations from the Majorana-Weyl to the standard representation

Implementing symmetries in the current model enables one to determine uniquely the form of the scattering matrices and of the propagation equations. Additionally, it provides a stability condition on the dynamics of the currents, which is fulfilled if the scattering matrices are unitary. Although this method was applied successfully in order to build the current model for the Majorana-Weyl construction (Sects. 2.3 and 2.4), it turns out that it fails for the standard representation. We thus proceed by using an indirect method in obtaining the \pm scattering matrices that relies on the linear transformation between the Majorana-Weyl and the standard representations. Contrary to the direct method, we show that there exist several possible expressions for the scattering matrices. The linear correspondence between the Majorana-Weyl and the standard representation reads

$$\begin{cases} \psi_+ = \psi_L + \psi_R, \\ \psi_- = \psi_L - \psi_R. \end{cases} \quad (83)$$

According to the definitions of the fields ψ_L and ψ_R (Eqs. (6, 7)), ψ_+ and ψ_- in (83) become

$$\psi_+ = \sum_{k=1}^3 \lambda_{k,L} E_{k,L} + \sum_{k=1}^3 \lambda_{k,R} E_{k,R}, \quad (84)$$

$$\psi_- = \sum_{k=1}^3 \lambda_{k,L} E_{k,L} - \sum_{k=1}^3 \lambda_{k,R} E_{k,R}. \quad (85)$$

If we remember that the $\lambda_{k,L}$ and the $\lambda_{k,R}$ are linked together by the relations (62), then it is possible to express ψ_+ and ψ_- as a function of the $\lambda_{k,L}$ or of the $\lambda_{k,R}$ uniquely. Let us first put the emphasis on the L parameters rather than on the R ones. For this purpose, let us

write equations (84, 85) in the following form

$$\begin{aligned} \psi_+ = & \frac{1}{2} \sum_{k=1,2} \left[(\lambda_{k,L} - i\varepsilon_\mu \lambda_{k,R})(E_{k,L} + i\varepsilon_\mu E_{k,R}) \right. \\ & \left. + (\lambda_{k,L} + i\varepsilon_\mu \lambda_{k,R})(E_{k,L} - i\varepsilon_\mu E_{k,R}) \right] \\ & + \frac{1}{2} \left[(\lambda_{3,L} + \lambda_{3,R})(E_{3,L} + E_{3,R}) \right. \\ & \left. + (\lambda_{3,L} - \lambda_{3,R})(E_{3,L} - E_{3,R}) \right], \end{aligned} \quad (86)$$

$$\begin{aligned} \psi_- = & \frac{1}{2} \sum_{k=1,2} \left[(\lambda_{k,L} + i\varepsilon_\mu \lambda_{k,R})(E_{k,L} + i\varepsilon_\mu E_{k,R}) \right. \\ & \left. + (\lambda_{k,L} - i\varepsilon_\mu \lambda_{k,R})(E_{k,L} - i\varepsilon_\mu E_{k,R}) \right] \\ & + \frac{1}{2} \left[(\lambda_{3,L} - \lambda_{3,R})(E_{3,L} + E_{3,R}) \right. \\ & \left. + (\lambda_{3,L} + \lambda_{3,R})(E_{3,L} - E_{3,R}) \right]. \end{aligned} \quad (87)$$

If we plug (62) into the above equations, we see that terms are cancelled in each equation so that ψ_+ and ψ_- become

$$\begin{aligned} \psi_+ = & \lambda_{1,L}(E_{1,L} - i\varepsilon_\mu E_{1,R}) + \lambda_{2,L}(E_{2,L} + i\varepsilon_\mu E_{2,R}) \\ & + \lambda_{3,L}(E_{3,L} + E_{3,R}), \end{aligned}$$

$$\begin{aligned} \psi_- = & \lambda_{1,L}(E_{1,L} + i\varepsilon_\mu E_{1,R}) + \lambda_{2,L}(E_{2,L} - i\varepsilon_\mu E_{2,R}) \\ & + \lambda_{3,L}(E_{3,L} - E_{3,R}). \end{aligned}$$

Now, identifying $\lambda_{k,+}$ and $\lambda_{k,-}$ with $\lambda_{k,L}$, $k = 1, 2, 3$, imposes the following relations between the \pm incident currents and the L, R ones

$$\begin{cases} E_{1,\pm} = E_{1,L} \mp i\varepsilon_\mu E_{1,R}, \\ E_{2,\pm} = E_{2,L} \pm i\varepsilon_\mu E_{2,R}, \\ E_{3,\pm} = E_{3,L} \pm E_{3,R}. \end{cases} \quad (88)$$

Assuming that the same relationships hold between the \pm and L, R outgoing currents, one finds that by substituting the propagation rules (Eqs. (4, 5)) for the L, R currents in (88), the propagation rules (66) for the \pm currents are recovered. The scattering matrices S_\pm are also deduced from (88), by using the expressions of the scattering matrices in the Majorana-Weyl representation (Eqs. (59, 60)) leading to

$$S_\pm = S_L. \quad (89)$$

If instead, we put the emphasis on the R currents rather than on the L ones, the \pm fields are obtained through the equations (86, 87) where the roles of the L and R subscripts are exchanged. If we plug (62) in those two equations, ψ_\pm becomes

$$\begin{aligned} \psi_+ = & \lambda_{1,R}(E_{1,R} + i\varepsilon_\mu E_{1,L}) + \lambda_{2,R}(E_{2,R} - i\varepsilon_\mu E_{2,L}) \\ & + \lambda_{3,R}(E_{3,R} + E_{3,L}), \end{aligned}$$

$$\begin{aligned} \psi_- = & -\lambda_{1,R}(E_{1,R} - i\varepsilon_\mu E_{1,L}) - \lambda_{2,R}(E_{2,R} + i\varepsilon_\mu E_{2,L}) \\ & - \lambda_{3,R}(E_{3,R} - E_{3,L}). \end{aligned}$$

Then, by choosing $\lambda_{k,+} = -\lambda_{k,-} = \lambda_{k,R}$, $k = 1, 2, 3$, we fix the relations between the \pm incident currents and the L, R ones

$$\begin{cases} E_{1,\pm} = E_{1,R} \pm i\varepsilon_\mu E_{1,L}, \\ E_{2,\pm} = E_{2,R} \mp i\varepsilon_\mu E_{2,L}, \\ E_{3,\pm} = E_{3,R} \pm E_{3,L}. \end{cases} \quad (90)$$

Again, if we assume that the same relationships hold between the outgoing currents of both representations, the propagation rules of the standard representation are recovered and the \pm scattering matrices read

$$S_\pm = S_R. \quad (91)$$

Finally, we present a more symmetric transformation between the two representations where the L and R currents are treated on the same footing. The starting point is to write the \pm fields (Eqs. (84, 85)) in the following form

$$\begin{aligned} \psi_+ = & \alpha^* \lambda_{1,L}(\alpha E_{1,L} + \alpha^* E_{1,R}) \\ & + \alpha \lambda_{2,L}(\alpha^* E_{2,L} + \alpha E_{2,R}) + \lambda_{3,L}(E_{3,L} + E_{3,R}), \end{aligned} \quad (92)$$

$$\begin{aligned} \psi_- = & \alpha \lambda_{1,L}(\alpha^* E_{1,L} + \alpha E_{1,R}) \\ & + \alpha^* \lambda_{2,L}(\alpha E_{2,L} + \alpha^* E_{2,R}) \\ & + \alpha^2 \lambda_{3,L}(\alpha^{*2} E_{3,L} + \alpha^2 E_{3,R}), \end{aligned} \quad (93)$$

where α is defined by $\alpha^2 = i\varepsilon_\mu$ (Eq. (58)). The identities: $\alpha^* \lambda_{1,L} = \alpha \lambda_{1,R}$, $\alpha \lambda_{1,L} = -\alpha^* \lambda_{1,R}$, $\alpha^* \lambda_{2,L} = -\alpha \lambda_{2,R}$, $\alpha \lambda_{2,L} = \alpha^* \lambda_{2,R}$, which follow from (62), insure that neither the L incident currents nor the R incident currents are preferred in equations (92, 93).

The choice

$$\begin{cases} \lambda_{1,+} = \alpha^* \lambda_{1,L}, & \lambda_{2,+} = \alpha \lambda_{2,L}, & \lambda_{3,+} = \lambda_{3,L}, \\ \lambda_{1,-} = \alpha \lambda_{1,L}, & \lambda_{2,-} = \alpha^* \lambda_{2,L}, & \lambda_{3,-} = \alpha^2 \lambda_{3,L}, \end{cases} \quad (94)$$

fixes ρ_+ and ρ_-

$$\begin{cases} \rho_{1,+} = \alpha \rho_{1,L}, & \rho_{2,+} = \alpha^* \rho_{2,L}, & \rho_{3,+} = \rho_{3,L}, \\ \rho_{1,-} = \alpha^* \rho_{1,L}, & \rho_{2,-} = \alpha \rho_{2,L}, & \rho_{3,-} = \alpha^{*2} \rho_{3,L}, \end{cases} \quad (95)$$

and the relations between the \pm incident currents and the L, R ones

$$\begin{cases} E_{1,+} = \alpha E_{1,L} + \alpha^* E_{1,R}, & E_{1,-} = \alpha^* E_{1,L} + \alpha E_{1,R}, \\ E_{2,+} = \alpha^* E_{2,L} + \alpha E_{2,R}, & E_{2,-} = \alpha E_{2,L} + \alpha^* E_{2,R}, \\ E_{3,+} = E_{3,L} + E_{3,R}, & E_{3,-} = \alpha^{*2} E_{3,L} + \alpha^2 E_{3,R}. \end{cases} \quad (96)$$

Assuming that the transformation rules (96) link also the outgoing currents of each representation, we recover the propagation rules of the standard representation and derive the \pm scattering matrices

$$S_\pm = -\frac{2 \sin \theta}{2 + |\nu|^2} \alpha^2 \begin{pmatrix} 1 & 1 & \alpha \nu \\ 1 & 1 & \alpha \nu \\ \alpha \nu & \alpha \nu & \alpha^2 \nu^2 \end{pmatrix} + \mu \text{diag}(\varepsilon_\mu, \varepsilon_\mu, 1), \quad (97)$$

$$S_- = -\frac{2 \sin \theta}{2 + |\nu|^2} \alpha^2 \begin{pmatrix} 1 & -1 & \alpha \nu \\ -1 & 1 & -\alpha \nu \\ \alpha \nu & -\alpha \nu & \alpha^2 \nu^2 \end{pmatrix} + \mu \text{diag}(\varepsilon_\mu, \varepsilon_\mu, 1). \quad (98)$$

Within this symmetric transformation both scattering matrices are different from S_L and S_R . Meanwhile, each one satisfies the symmetries : $S_\pm = {}^T S_\pm$, $S_\pm^{-1} = S_\pm^*$, and are indeed unitary matrices. The first property, namely $S_\pm = {}^T S_\pm$, can be related to a reciprocity invariance of the scattering process, whether the second property, $S_\pm^{-1} = S_\pm^*$, can be related to a time reversal invariance of the scattering process. But we stress that neither this reciprocity property nor this time reversal symmetry are symmetries obeyed by the components ψ_+ and ψ_- of the spinor field in the standard representation. Hence, contrary to the Majorana-Weyl representation, the symmetries of the current scattering process are different from the symmetries underlying the Dirac equation in the standard representation. This explains why we did not succeed in the direct construction of the standard representation.

Finally, using equations (94, 95), equations (80, 81) become

$$\begin{aligned} & \frac{1}{2\tau} (\psi_+(n, t + \tau) - \psi_+(n, t - \tau)) = \\ & \left(\frac{2 \sin \theta}{2 + |\nu|^2} \right) \frac{a}{\tau} \left(\frac{\psi_-(n + a, t) - \psi_-(n - a, t)}{2a} \right) \\ & - \frac{i}{\tau} \left(\frac{\sin \theta |\nu|^2}{2 + |\nu|^2} \right) \psi_+(n, t), \end{aligned} \quad (99)$$

$$\begin{aligned} & \frac{1}{2\tau} (\psi_-(n, t + \tau) - \psi_-(n, t - \tau)) = \\ & \left(\frac{2 \sin \theta}{2 + |\nu|^2} \right) \frac{a}{\tau} \left(\frac{\psi_+(n + a, t) - \psi_+(n - a, t)}{2a} \right) \\ & + \frac{i}{\tau} \left(\frac{\sin \theta |\nu|^2}{2 + |\nu|^2} \right) \psi_-(n, t), \end{aligned} \quad (100)$$

where

$$\begin{cases} \mu^2 \left(1 + \sum_{k=1}^3 \frac{\lambda_{k,+} \rho_{k,+}}{\mu_{k,+}} \right) = \mu^2 \left(1 + \sum_{k=1}^3 \frac{\lambda_{k,-} \rho_{k,-}}{\mu_{k,-}} \right) = 1, \\ \lambda_{2,+} \rho_{1,-} = \lambda_{2,-} \rho_{1,+} = -\frac{2 \sin \theta}{2 + |\nu|^2}, \\ \lambda_{1,+} \rho_{2,-} = \lambda_{1,-} \rho_{2,+} = \frac{2 \sin \theta}{2 + |\nu|^2}, \\ \lambda_{3,+} \rho_{3,+} = \lambda_{3,-} \rho_{3,-} = -i \frac{2 \sin \theta |\nu|^2}{2 + |\nu|^2}. \end{cases} \quad (101)$$

By using the relations (56, 57) one can check that equations (99, 100) are the discretized version of the Dirac equation in the standard representation.

4 Conclusion and perspectives

In this article we have recovered the Dirac equation in 1 + 1 space-time dimensions from a microscopic current model based on the Huygens principle and constructed from suitable symmetries, thus extending to spinor waves a previous work describing the propagation of scalar waves in an arbitrary inhomogeneous medium [6]. This is not surprising since the current model combines all the basic features of wave propagation and since the symmetries are those underlying the Dirac equation. The originality of the method stems from the systematic construction based on very simple dynamical rules. Moreover the equation derived in this paper is independent of the type of discrete lattice and could be derived on any graph. The method developed in this paper is very close to the philosophy sustaining the construction of various cellular automata which are routinely used to solved numerically hydrodynamic or phase transition problems. Indeed, even if we cannot assign a precise physical meaning to the current variables, we choose the dynamics of those basic variables according to the fundamental laws of wave propagation. Hopefully, the current model for the Dirac equation could be used as a useful tool to study numerically the time-dependent spinor propagation in complex systems with any kind of boundary conditions.

A natural extension of this work would be to describe spinor field propagation in inhomogeneous media. Indeed, a wide range of condensed matter problems are related to the random mass Dirac equation in 1 + 1 space-time dimensions and could be studied by using such a model. Random mass Dirac spinors show up in the problem of a one-dimensional metal with a half-filled electron band and random backscattering [17], spin-ladder or quasi-one-dimensional spin Peierls systems [18]. In order to derive a Dirac equation with a mass and a velocity varying with the space coordinate, we can use the method developed in [6]. The description of scalar waves in an inhomogeneous medium was achieved with the help of additional currents that trap a fraction of the “energy” on each node. Such additional currents can also be introduced for the Dirac equation.

Another extension of this work would be to construct the Dirac equation in higher space dimensions. This should be useful for instance to study the plateau transitions in the integer quantum Hall effect where the Dirac equation shows up in 2 + 1 space-time dimensions [19]. Though the calculations are rather cumbersome, the Dirac equation in 3 + 1 space-time dimensions can also be constructed with our method.

Finally, it would be interesting to see whether the current model described in this paper could be useful as a starting point for a path-integral formulation of the usual Dirac equation in 1 + 1 space-time dimensions [20]. Despite the fact that the formulation of path integrals for

$$\begin{aligned} \frac{i}{2\tau} (\psi_L(n, t + \tau) - \psi_L(n, t - \tau)) = \\ - \left(\frac{\sin \theta}{2 + |\nu|^2} \right) \frac{a^2}{\tau} \left(\frac{\psi_L(n + a, t) + \psi_L(n - a, t) - 2\psi_L(n, t)}{a^2} \right) + \frac{\varepsilon_\gamma}{\tau} \left(\frac{\sin \theta |\nu|^2}{2 + |\nu|^2} \right) \psi_R(n, t) - \frac{1}{\tau} \left(\frac{2 \sin \theta}{2 + |\nu|^2} \right) \psi_L(n, t), \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \frac{i}{2\tau} (\psi_R(n, t + \tau) - \psi_R(n, t - \tau)) = \\ - \left(\frac{\sin \theta}{2 + |\nu|^2} \right) \frac{a^2}{\tau} \left(\frac{\psi_R(n + a, t) + \psi_R(n - a, t) - 2\psi_R(n, t)}{a^2} \right) + \frac{\varepsilon_\gamma}{\tau} \left(\frac{\sin \theta |\nu|^2}{2 + |\nu|^2} \right) \psi_L(n, t) - \frac{1}{\tau} \left(\frac{2 \sin \theta}{2 + |\nu|^2} \right) \psi_R(n, t), \end{aligned} \quad (\text{A.3})$$

spinor fields in terms of non-commuting variables is not new within the framework of quantum field theory, this description suffers from a lack of physical interpretation. As an example the field-theoretical description of a spinor field doesn't describe the way the fermion particle moves along a selected path as sketched by Feynman [21]. Some progresses in this direction have been done by extending the analogy between quantum mechanics and brownian motion to the Dirac equation [22–25]. But this point of view focusses on a probabilistic interpretation of the Dirac particle through Poisson processes. On the contrary our model deals with the Dirac equation itself and should give a direct interpretation of the paths followed by a spinor particle between two points.

During the preparation of this work, we have benefited from discussions with D. Sornette. S. DTA acknowledges the hospitality of the Earth and Space Sciences at UCLA and wants also to thank warmly Jean-Louis Pichard, at the Service de Physique de l'État Condensé du CE de Saclay, and Jean-Marc Luck, at the Service de Physique Théorique du CE de Saclay, for hospitality when the paper has been completed. We are also indebted to Jean-Marc Luck for a careful reading of the manuscript.

Appendix: Construction of the Schrödinger equation with a spin

This appendix deals with the construction of a Schrödinger equation for a quantum particle with a spin 1/2 in 1 + 1 space-time dimensions. This equation reads

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + v' \right) Id + v\sigma_1 \right] \Psi, \quad (\text{A.1})$$

where $\Psi = (\psi_L \ \psi_R)^T$ is a two-component field. In Section 2.4, we have obtained two discretized propagation equations for the chiral fields (Eqs. (54, 55)) where the order of the discrete spatial derivative is controlled by the ratio $(\lambda_{2,L}/\lambda_{1,L})^*/(\lambda_{2,L}/\lambda_{1,L})$. To derive the Dirac equation, we have chosen $\lambda_{2,L}/\lambda_{1,L}$ to be imaginary in order to obtain a first order spatial derivative. If we assign $\lambda_{2,L}/\lambda_{1,L}$ to be real then a second order spatial derivative arises. We write $\lambda_{2,L}/\lambda_{1,L} = \varepsilon_\lambda$ where $\varepsilon_\lambda = \pm 1$. The propagation equations (54, 55) depend now on the

two signs $\varepsilon \equiv \varepsilon_\lambda \varepsilon_\mu, \varepsilon_\gamma$, and on the two real parameters $|\nu|, \theta$. Then, we recognize the discretized version of the Schrödinger equation with a spin (A.1) if we fix $\varepsilon = -1$ in the two wave propagation equations

see equations (A.2) and (A.3) above

provided that

$$\frac{\hbar}{2m} = \left(\frac{\sin \theta}{2 + |\nu|^2} \right) \frac{a^2}{\tau} = -\frac{a^2 v'}{2 \hbar}, \quad (\text{A.4})$$

$$\frac{v}{\hbar} = \left(\frac{\sin \theta |\nu|^2}{2 + |\nu|^2} \right) \frac{\varepsilon_\gamma}{\tau}. \quad (\text{A.5})$$

The L, R scattering matrices are parametrized by three real parameters: ε_μ, θ and $|\nu|$, as ν is a purely imaginary number (using (47, 48)). Introducing again α (Eq. (58)), the matrices read

$$S_L = -\frac{2 \sin \theta}{2 + |\nu|^2} \alpha^2 \begin{pmatrix} 1 & i\alpha^2 & \nu \\ i\alpha^2 & 1 & i\alpha^2 \nu \\ i\alpha^2 \nu & \nu & i\alpha^2 \nu^2 \end{pmatrix} + \mu \text{diag}(\varepsilon_\mu, \varepsilon_\mu, 1), \quad (\text{A.6})$$

$$S_R = -\frac{2 \sin \theta}{2 + |\nu|^2} \alpha^2 \begin{pmatrix} 1 & i\alpha^2 & i\alpha^2 \nu \\ i\alpha^2 & 1 & \nu \\ \nu & i\alpha^2 \nu & i\alpha^2 \nu^2 \end{pmatrix} + \mu \text{diag}(\varepsilon_\mu, \varepsilon_\mu, 1). \quad (\text{A.7})$$

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